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Erratum to: A \mathbb{Q} -factorial complete toric variety is a quotient of a poly weighted space

Michele Rossi · Lea Terracini

After the publication of [2], we realized that Proposition 3.1, in that paper, contains an error, whose consequences are rather pervasive along the whole section 3 and for some aspects of examples 5.1 and 5.2. Here we give a complete account of needed corrections.

First of all [2, Prop. 3.1] has to be replaced by the following:

PROPOSITION 3.1 *Let $X(\Sigma)$ be a \mathbb{Q} -factorial complete toric variety and $Y(\widehat{\Sigma})$ be its universal 1-covering. Let $\{D_\rho\}_{\rho \in \Sigma(1)}$ and $\{\widehat{D}_\rho\}_{\rho \in \widehat{\Sigma}(1)}$ be the standard bases of $\mathcal{W}_T(X)$ and $\mathcal{W}_T(Y)$, respectively, given by the torus orbit closures of the rays. Then*

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \in \mathcal{C}_T(X) \implies \widehat{D} = \sum_{\rho \in \widehat{\Sigma}(1)} a_\rho \widehat{D}_\rho \in \mathcal{C}_T(Y).$$

Therefore, under the identification $\mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X) \xrightarrow{\alpha} \mathcal{W}_T(Y) \cong \mathbb{Z}^{|\widehat{\Sigma}(1)|}$ realized by the isomorphism $D_\rho \xrightarrow{\alpha} \widehat{D}_\rho$,

$$\mathcal{C}_T(X) \cong \alpha(\mathcal{C}_T(X)) \leq \mathcal{C}_T(Y) \leq \mathcal{W}_T(Y)$$

is a chain of subgroup inclusions. Moreover the induced morphism $\bar{\alpha} : \text{Cl}(X) \rightarrow \text{Cl}(Y)$ is injective when restricted to $\text{Pic}(X)$, realizing the following further chain of subgroup inclusions

$$\text{Pic}(X) \cong \bar{\alpha}(\text{Pic}(X)) \leq \text{Pic}(Y) \leq \text{Cl}(Y)$$

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Michele Rossi

Dipartimento di Matematica - Universit  di Torino - Via Carlo Alberto 10 - 10123 Torino (Italy)

Tel.: +39-011-6702813

Fax: +39-011-6702878

E-mail: michele.rossi@unito.it

Lea Terracini

Dipartimento di Matematica - Universit  di Torino - Via Carlo Alberto 10 - 10123 Torino (Italy)

Tel.: +39-011-6702813

Fax: +39-011-6702878

E-mail: lea.terracini@unito.it

Proof: Let us fix a basis \mathcal{B} of the \mathbb{Z} -module $M \cong \mathbb{Z}^n$ and let V and \widehat{V} be fan matrices representing the standard morphisms

$$\mathrm{div}_X : M \cong \mathbb{Z}^n \xrightarrow{V^T} \mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X) \quad , \quad \mathrm{div}_Y : M \cong \mathbb{Z}^r \xrightarrow{\widehat{V}^T} \mathbb{Z}^{|\widehat{\Sigma}(1)|} \cong \mathcal{W}_T(Y)$$

Let $\beta \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$ be such that $V = \beta \widehat{V}$ and so realizing an injective endomorphism of the \mathbb{Z} -module M . The result follows by writing down the condition of being locally principal for a Weil divisor and observing that

$$\begin{aligned} \mathcal{I}^\Sigma &= \{I \subseteq \{1, \dots, n+r\} : \langle V^I \rangle \in \Sigma(n)\} \\ &= \{I \subseteq \{1, \dots, n+r\} : \langle \widehat{V}^I \rangle \in \widehat{\Sigma}(n)\} = \mathcal{I}^{\widehat{\Sigma}} \end{aligned} \quad (1)$$

by the construction of $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$, given the choice of $\Sigma \in \mathcal{SF}(V)$. Notice that \mathcal{I}^Σ describes the complements of those sets described by \mathcal{I}_Σ , as defined in [2, Rem.2.4]. In particular the Weil divisor $\sum_{j=1}^{n+r} a_j D_j \in \mathcal{W}_T(X)$ is Cartier if and only if

$$\forall I \in \mathcal{I}^\Sigma \quad \exists \mathbf{m}_I \in M : \forall j \notin I \quad \mathbf{v}_j^T \mathbf{m}_I = a_j, \quad (2)$$

where \mathbf{v}_j is the j -th column of V . Then $\alpha(\sum_{j=1}^{n+r} a_j D_j) = \sum_{j=1}^{n+r} a_j \widehat{D}_j$ is a Cartier divisor since

$$\forall I \in \mathcal{I}^\Sigma \quad \forall j \notin I \quad \widehat{\mathbf{v}}_j^T (\beta^T \mathbf{m}_I) = a_j$$

where $\widehat{\mathbf{v}}_j$ is the j -th column of \widehat{V} .

The injectivity of $\bar{\alpha}$ follows from the well-known freeness of $\mathrm{Pic}(X)$. \square

As a consequence, parts 1, 4, 5 of [2, Thm. 3.2] still hold, while parts 2, 3, 6, 7 have to be replaced by the following:

THEOREM 3.2 *Let $X = X(\Sigma)$ be a n -dimensional \mathbb{Q} -factorial complete toric variety of rank r and $Y = Y(\widehat{\Sigma})$ be its universal 1-covering. Let V be a reduced fan matrix of X , $Q = \mathcal{G}(V)$ a weight matrix of X and $\widehat{V} = \mathcal{G}(Q)$ be a CF -matrix giving a fan matrix of Y .*

2. *Define \mathcal{I}^Σ as in (1). For any $I \in \mathcal{I}^\Sigma$ let E_I be the $r \times (n+r)$ matrix admitting as rows the standard basis vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, for $i \in I$, representing the i -th basis divisor $D_i \in \mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$. Set $\widetilde{V}_I := (V^T \mid E_I^T) \in \mathbf{M}_{n+r}(\mathbb{Z})$. Then Cartier divisors give rise to the following maximal rank subgroup of $\mathcal{W}_T(X)$*

$$\mathcal{C}_T(X) \cong \bigcap_{I \in \mathcal{I}^\Sigma} \mathcal{L}_c(\widetilde{V}_I) \leq \mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X)$$

and a basis of $\mathcal{C}_T(X) \leq \mathcal{W}_T(X)$ can be explicitly computed by applying the procedure described in [1, § 1.2.3].

3. *Let $C_X \in \mathrm{GL}_{n+r}(\mathbb{Q}) \cap \mathbf{M}_{n+r}(\mathbb{Z})$ be a matrix whose rows give a basis of $\mathcal{C}_T(X)$ in $\mathcal{W}_T(X)$, as obtained in the previous part 2. Identify $\mathrm{Cl}(X)$ with $\mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k \mathbb{Z}$ by item (c) of part 4 in [2, Thm. 3.2], and represent the morphism d_X by $Q \oplus \Gamma$, according to parts 1 and 5. Let $A \in \mathrm{GL}_{n+r}(\mathbb{Z})$ be a matrix such that $A \cdot C_X \cdot Q^T$ is in HNF. Let $\mathbf{c}_1, \dots, \mathbf{c}_r$ be the first r rows of the matrix $A \cdot C_X$ and for $i = 1, \dots, r$ put $\mathbf{b}_i = Q \cdot \mathbf{c}_i^T + \Gamma \cdot \mathbf{c}_i^T$. Then $\mathbf{b}_1, \dots, \mathbf{b}_r$ is a basis of the free group $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$.*
6. *Given the choice of \widehat{V} and V as in the previous parts 4 and 5 of [2, Thm. 3.2], consider*

$$U := \begin{pmatrix} rU_Q \\ \widehat{V} \end{pmatrix} \in \mathrm{GL}_{n+r}(\mathbb{Z})$$

$$W \in \mathrm{GL}_{n+r}(\mathbb{Z}) : W \cdot ({}^{n+r-s}U)^T = \mathrm{HNF} \left(({}^{n+r-s}U)^T \right)$$

$$G := {}_s\widehat{V} \cdot ({}_sW)^T \in \mathbf{M}_s(\mathbb{Z})$$

$$U_G \in \mathrm{GL}_s(\mathbb{Z}) : U_G \cdot G^T = \mathrm{HNF}(G^T).$$

Then a “torsion matrix” representing the “torsion part” of the morphism d_X , that is, $\tau_X : \mathcal{W}_T(X) \rightarrow \text{Tors}(\text{Cl}(X))$, is given by

$$\Gamma = U_G \cdot {}_s W \pmod{\tau} \quad (3)$$

where this notation means that the (k, j) -entry of Γ is given by the class in $\mathbb{Z}/\tau_k \mathbb{Z}$ represented by the corresponding (k, j) -entry of ${}^s U_G \cdot {}_s W$, for every $1 \leq k \leq s$, $1 \leq j \leq n+r$.

7. Setting $\delta_\Sigma := \text{lcm}(\det(Q_I) : I \in \mathcal{I}^\Sigma)$ then

$$\delta_\Sigma \mathcal{W}_T(X) \subseteq \mathcal{C}_T(X) \quad \text{and} \quad \delta_\Sigma \mathcal{W}_T(Y) \subseteq \mathcal{C}_T(Y)$$

and there are the following divisibility relations

$$\delta_\Sigma \mid [\text{Cl}(Y) : \text{Pic}(Y)] = [\mathcal{W}_T(Y) : \mathcal{C}_T(Y)] \mid [\text{Cl}(X) : \text{Pic}(X)] = [\mathcal{W}_T(X) : \mathcal{C}_T(X)].$$

Proof: (2): Recalling relation (2) in the proof of Proposition 3.1, set

$$\forall I \in \mathcal{I}^\Sigma \quad \mathcal{P}^I = \{L = \sum_{j=1}^{n+r} a_j D_j \in \mathcal{W}_T(X) \mid \exists \mathbf{m} \in M : \forall j \notin I \quad \mathbf{m} \cdot \mathbf{v}_j = a_j\}.$$

Then \mathcal{P}^I contains $\text{Im}(\text{div}_X : M \rightarrow \mathcal{W}_T(X)) = \mathcal{L}_c(V^T)$ and a \mathbb{Z} -basis of \mathcal{P}^I is given by

$$\{D_j, j \in I\} \cup \left\{ \sum_{k=1}^{n+r} v_{ik} D_k, i = 1, \dots, n \right\},$$

where $\{v_{ik}\}$ is the i -th entry of \mathbf{v}_k , so giving the rows of the matrix \tilde{V}_I defined in the statement.

(3): By definition

$$\text{Pic}(X) = \text{Im}(\mathcal{C}_T(X) \hookrightarrow \mathcal{W}_T(X) \xrightarrow{d_X} \text{Cl}(X))$$

so that $\text{Pic}(X)$ is generated by the image under $Q \oplus \Gamma$ of the transposed of the rows of C_X . Since $\text{rk}(C_X) = n+r$ and $\text{rk}(Q) = r$, the matrix $C_X \cdot Q^T$ has rank r and therefore its HNF has the last $n-r$ rows equal to zero. Therefore the rows of the matrix $A \cdot C_X$ provide a basis of $\mathcal{C}_T(X)$ in $\mathcal{W}_T(X)$ such that its last n rows are a basis of $\mathcal{L}_r(\hat{V}) \cap \mathcal{C}_T(X) = \mathcal{L}_r(V)$. Since $\text{Pic}(X)$ is free of rank r it is freely generated by the images under d_X of the first r rows.

(6): A representative matrix of the torsion part $\tau_X : \mathcal{W}_T(X) \rightarrow \text{Cl}(X)$ of the morphism d_X is any matrix satisfying the following properties:

- (i) $\Gamma = (\gamma_{kj})$ with $\gamma_{kj} \in \mathbb{Z}/\tau_k \mathbb{Z}$,
- (ii) $\Gamma \cdot ({}^t U_Q)^T = \mathbf{0}_{s,r} \pmod{\tau}$, meaning that Γ kills the generators of the free part $F \leq \text{Cl}(X)$ defined in display (4) of part 1 of [2, Thm. 3.2],
- (iii) $\Gamma \cdot V^T = \mathbf{0}_{s,n} \pmod{\tau}$, where V is a fan matrix satisfying condition 4.(b) in [2, Thm. 3.2]: this is due to the fact that the rows of V span $\ker(d_X)$,
- (iv) $\Gamma \cdot ({}_s \hat{V})^T = \mathbf{I}_s \pmod{\tau}$, since the rows of ${}_s \hat{V}$ give the generators of $\text{Tors}(\text{Cl}(X))$, as in display (6) of part 5 of [2, Thm. 3.2].

Therefore it suffices to show that the matrix $U_G \cdot {}_s W$ in (3) satisfies the previous conditions (ii), (iii) and (iv) without any reduction mod τ , that is,

$$U_G \cdot {}_s W \cdot ({}^{n+r-s} U)^T = \mathbf{0}_{s, n+r-s} \quad , \quad U_G \cdot {}_s W \cdot ({}_s \hat{V})^T = \mathbf{I}_s .$$

The first equation follows by the definition of W , in fact

$$W \cdot ({}^{n+r-s} U)^T = \text{HNF} \left(({}^{n+r-s} U)^T \right) = \begin{pmatrix} \mathbf{I}_{n+r-s} \\ \mathbf{0}_{s, n+r-s} \end{pmatrix} \Rightarrow {}_s W \cdot ({}^{n+r-s} U)^T = \mathbf{0}_{s, n+r-s}$$

The second equation follows by the definition of U_G , in fact

$$U_G \cdot {}_s W \cdot ({}_s \widehat{V})^T = U_G \cdot G^T = \text{HNF}(G^T) = \mathbf{I}_s.$$

(7): Part (4) of [1, Thm. 2.9] gives that $\delta_\Sigma \mid [\text{Cl}(Y) : \text{Pic}(Y)] = [\mathcal{W}_T(Y) : \mathcal{C}_T(Y)]$. On the other hand Proposition 3.1 gives that $[\mathcal{W}_T(Y) : \mathcal{C}_T(Y)] \mid [\mathcal{W}_T(X) : \mathcal{C}_T(X)] = [\text{Cl}(X) : \text{Pic}(X)]$. \square

Considerations i, ii, iii, iv, v of [2, Rem. 3.3] still holds, while vi, vii and the remaining part of Remark 3.3 have to be replaced by the following

REMARK 3.3

- vi. apply procedure [1, § 1.2.3], based on the HNF algorithm, to get a $(n+r) \times (n+r)$ matrix C_X whose rows give a basis of $\mathcal{C}_T(X) \leq \mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$;
- vii. apply procedure described in part 6 of Theorem 3.2 to get a system of generators of $\text{Pic}(X)$ in $\text{Cl}(X)$. Precisely, let $A \in \text{GL}_{n+r}(\mathbb{Z})$ be a switching matrix such that $\text{HNF}(C_X \cdot Q^T) = A \cdot C_X \cdot Q^T$, and put

$$B_X = {}^r(A \cdot C_X \cdot Q^T), \quad \Theta_X = {}^r(A \cdot C_X \cdot \Gamma^T) \quad (4)$$

Then the rows of the matrices B_X and Θ_X represent respectively the free part and the torsion part of a basis of $\text{Pic}(X)$ in $\text{Cl}(X)$, where the latter is identified to $\mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k \mathbb{Z}$.

Moreover:

- recall that, for the universal 1-covering Y of X , once fixed the basis $\{\widehat{D}_j\}_{j=1}^{n+r}$ of $\mathcal{W}_T(Y) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_Y(\widehat{L}_i)\}_{i=1}^r$ of $\text{Cl}(Y) \cong \mathbb{Z}^r$, (see (11) in [1, Thm. 2.9]), one gets the following commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & M & \xrightarrow{\begin{pmatrix} \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix}} & \mathcal{C}_T(Y) \cong \text{Pic}(Y) \oplus M & \xrightarrow{\begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r,n} \end{pmatrix}} & \text{Pic}(Y) & \longrightarrow 0 \\
 & \parallel & & \downarrow C_Y^T & & \downarrow B_Y^T & \\
 0 \longrightarrow & M & \xrightarrow[\widehat{V}^T]{\text{div}_Y} & \mathcal{W}_T(Y) = \bigoplus_{j=1}^{n+r} \mathbb{Z} \cdot D_j & \xrightarrow[Q]{d_Y} & \text{Cl}(Y) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow & \mathcal{T}_Y & \xrightarrow{\cong} & \mathcal{T}_Y & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
 \end{array}$$

where B_Y is the $r \times r$ matrix constructed in [1, Thm. 2.9(3)] and

$$C_Y = \begin{pmatrix} B_Y & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix} \cdot U_Q = \begin{pmatrix} B_Y \cdot {}^r U_Q \\ \widehat{V} \end{pmatrix},$$

- once fixed the basis $\{D_j\}_{j=1}^{n+r}$ for $\mathcal{W}_T(X) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_X(L_i)\}_{i=1}^r$ of the free part $F \cong \mathbb{Z}^r$ of $\text{Cl}(X)$, constructed in part 1 of [2, Thm. 3.2], one gets the following commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & M & \xrightarrow{(\mathbf{0}_{n,r} \mid \mathbf{I}_n)} & \mathcal{C}_T(X) \cong \text{Pic}(X) \oplus M & \xrightarrow{(\mathbf{I}_r \mid \mathbf{0}_{r,n})} & \text{Pic}(X) & \longrightarrow 0 \\
 & \parallel & & \downarrow C_X^T & & \downarrow B_X^T \oplus \Theta_X^T & \\
 0 \longrightarrow & M & \xrightarrow[\mathbf{V}^T]{\text{div}_X} & \mathcal{W}_T(X) = \bigoplus_{j=1}^{n+r} \mathbb{Z} \cdot D_j & \xrightarrow[\mathbf{Q} \oplus \mathbf{F}]{d_X = f_X \oplus \tau_X} & \text{Cl}(X) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow & \mathcal{T}_X & \xrightarrow{\cong} & \mathcal{T}_X & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
 \end{array}$$

Moreover:

- recall the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & & & & & \ker(\bar{\alpha}) = \text{Tors}(\text{Cl}(X)) & \\
 0 \longrightarrow & M & \xrightarrow[\mathbf{V}^T]{\text{div}_X} & \mathcal{W}_T(X) = \mathbb{Z}^{|\Sigma(1)|} & \xrightarrow{d_X} & \text{Cl}(X) & \longrightarrow 0 \\
 & \downarrow \beta^T & & \downarrow \mathbf{I}_{n+r} \alpha & & \downarrow \bar{\alpha} & \\
 0 \longrightarrow & M & \xrightarrow[\hat{\mathbf{V}}^T]{\text{div}_Y} & \mathcal{W}_T(Y) = \mathbb{Z}^{|\hat{\Sigma}(1)|} & \xrightarrow{d_Y} & \text{Cl}(Y) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \text{coker}(\beta^T) \cong \text{Tors}(\text{Cl}(X)) & & 0 & & 0 & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
 \end{array}$$

(5)

[illegible]
$$\begin{aligned} \text{coker}(\beta^T) &\cong \ker(\bar{\alpha}) \cong \text{Tors}(\text{Cl}(X)) \\ \mathcal{K} &\cong \text{coker}(\alpha_{\mid}) \cong \mathcal{C}_T(Y)/\mathcal{C}_T(X) \end{aligned}$$
$$\begin{array}{ccccccc}
& & & 0 & & & (7) \\
& & & \downarrow & & & \\
0 & \longrightarrow & \text{Tors}(\text{Cl}(X)) & \longrightarrow & \mathcal{C}_T(Y)/\mathcal{C}_T(X) & \longrightarrow & \text{Pic}(Y)/\text{Pic}(X) \longrightarrow 0 \\
& & & & \downarrow & & \\
& & & & \text{Cl}(X)/\text{Pic}(X) & & \\
& & & & \downarrow & & \\
& & & & \text{Cl}(Y)/\text{Pic}(Y) & & \\
& & & & \downarrow & & \\
& & & & 0 & &
\end{array}$$

For what concerns the examples given in section 5, considerations related with parts v, vi and vii of Remark 3.3 have to be replaced as follows

EXAMPLE 5.1

v. A matrix $W \in \mathrm{GL}_4(\mathbb{Z})$ such that $\mathrm{HNF}(({}^3U)^T) = W \cdot ({}^3U)^T$ is given by

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

giving

$$G := {}_1\hat{V} \cdot ({}_1W)^T = \begin{pmatrix} 1 \end{pmatrix}$$

Therefore

$$\Gamma = {}_1W \pmod{5} = \begin{pmatrix} [0]_5 & [4]_5 & [2]_5 & [1]_5 \end{pmatrix}.$$

Consequently display (16) in [2], giving the action of $\mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), \mathbb{C}^*) \cong \mu_5$ on $Y = \mathbb{P}^3$, should be replaced by the following (equivalent) one:

$$\begin{aligned} \mu_5 \times \mathbb{P}^3 &\longrightarrow \mathbb{P}^3 \\ (\varepsilon, [x_1 : \dots : x_4]) &\mapsto [x_1 : \varepsilon^4 x_2 : \varepsilon^2 x_3 : \varepsilon x_4] \end{aligned} \quad (8)$$

vi. Applying procedure [1, § 1.2.3] as described in part 2 of Theorem 3.2, one gets a 4×4 matrix C_X whose rows give a basis of $\mathcal{C}_T(X)$ inside $\mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$. Namely

$$C_X = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ -3 & -3 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{pmatrix}$$

meaning that

$$\mathcal{C}_T(X) = \mathcal{L}(5D_1, 5D_2, -3D_1 - 3D_2 + D_3, -2D_1 - 4D_2 + D_4).$$

On the other hand, by part (3) of [1, Thm. 2.9], a basis of $\mathcal{C}_T(Y) \subseteq \mathcal{W}_T(Y)$ is given by the rows of

$$C_Y = \mathbf{I}_4 \cdot U_Q = U_Q \in \mathrm{GL}_n(\mathbb{Z})$$

giving $\mathcal{C}_T(Y) = \mathcal{W}_T(Y)$, as expected for $Y = \mathbb{P}^3$.

vii. A basis of $\mathrm{Pic}(X)$ inside $\mathrm{Cl}(X)$ is then obtained by applying part 6 of Theorem 3.2. With the notation of Remark 3.3 vii, a switching matrix A such that $A \cdot C_X \cdot Q^T$ is in HNF is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$\begin{aligned} B_X &= {}^1(A \cdot C_X \cdot Q^T) = \begin{pmatrix} 5 \end{pmatrix} \\ \Theta_X &= {}^1(A \cdot C_X \cdot \Gamma^T) = \begin{pmatrix} 0 \end{pmatrix} \end{aligned}$$

Then

$$\mathrm{Pic}(X) \cong \mathbb{Z}[5d_X(D_1)] \leq \mathbb{Z}[d_X(D_1)] \oplus \mathbb{Z}/5\mathbb{Z}[d_X(D_3 - D_4)] \cong \mathrm{Cl}(X) \Rightarrow \mathrm{Cl}(X)/\mathrm{Pic}(X) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}.$$

EXAMPLE 5.2

v. A matrix U as defined in part 6 of Theorem 3.2 is given by

$$U = \begin{pmatrix} {}^2U_Q \\ \widehat{V}' \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -6 & 3 & 1 & 0 & 0 & 0 \\ 521 & -251 & -168 & -2 & 14 & 28 \\ 388 & -222 & -112 & 7 & 45 & 3 \\ -184 & 105 & 53 & -2 & -23 & -1 \\ 191 & -109 & -55 & 2 & 24 & 1 \end{pmatrix}$$

A matrix $W \in \mathrm{GL}_6(\mathbb{Z})$ such that $\mathrm{HNF}({}^4U^T) = W \cdot ({}^4U)^T$ is given by

$$W = \begin{pmatrix} -57 & -115 & 3 & -549 & 17 & 0 \\ 4 & 8 & 1 & 3 & 7 & 0 \\ -125 & -250 & 0 & -1090 & 14 & 0 \\ -170 & -340 & 0 & -1482 & 19 & 0 \\ -188 & -376 & 0 & -1639 & 21 & 0 \\ -126 & -252 & 0 & -1092 & 13 & 1 \end{pmatrix}$$

then

$$G = {}_2\widehat{V}' \cdot ({}_2W)^T = \begin{pmatrix} -2093 & -1392 \\ 2302 & 1531 \end{pmatrix}$$

A matrix $U_G \in \mathrm{GL}_2(\mathbb{Z})$ such that $\mathrm{HNF}(G^T) = U_G \cdot G^T$ is given by

$$U_G = \begin{pmatrix} 1531 & -2302 \\ 1392 & -2093 \end{pmatrix}$$

hence giving

$$\begin{aligned} \Gamma &= U_G \cdot {}_2W \pmod{\tau} \\ &= \begin{pmatrix} 2224 & 4448 & 0 & 4475 & 2225 & -2302 \\ 2022 & 4044 & 0 & 4068 & 2023 & -2093 \end{pmatrix} \pmod{\begin{pmatrix} 3 \\ 15 \end{pmatrix}} \\ &= \begin{pmatrix} [1]_3 & [2]_3 & [0]_3 & [2]_3 & [2]_3 & [2]_3 \\ [12]_{15} & [9]_{15} & [0]_{15} & [3]_{15} & [13]_{15} & [7]_{15} \end{pmatrix} \end{aligned}$$

Consequently display (20) in [2] should be replaced by the following (equivalent) one

$$\begin{aligned} g(((t_1, t_2), \varepsilon, \eta), (x_1, \dots, x_6)) &:= \\ \left(t_1^2 t_2 \varepsilon \eta^{12} x_1, t_1^4 t_2 \varepsilon^2 \eta^9 x_2, t_1 t_2^3 x_3, t_1^5 t_2^2 \varepsilon^2 \eta^3 x_4, t_1^4 t_2^3 \varepsilon^2 \eta^{13} x_5, t_1^3 t_2^7 \varepsilon^2 \eta^7 x_6 \right) \end{aligned} \quad (9)$$

vi. Depending on the choice of the fan $\Sigma_i \in \mathcal{SF}(V)$, by applying procedure [1, § 1.2.3] as described in part 2 of Theorem 3.2, one gets a 6×6 matrix $C_{X,i}$ whose rows give a basis of $\mathcal{C}_T(X_i)$ inside $\mathcal{W}_T(X_i) \cong \mathbb{Z}^{|\Sigma_i(1)|}$. Namely

$$C_{X,1} = \begin{pmatrix} 265926375 & 0 & 0 & 0 & 0 & 0 \\ -148978500 & 825 & 0 & 0 & 0 & 0 \\ -58474020 & -375 & 15 & 0 & 0 & 0 \\ 37 & -18 & -7 & 1 & 0 & 0 \\ -58473933 & -417 & -3 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{X,2} = \begin{pmatrix} 43543500 & 0 & 0 & 0 & 0 & 0 \\ -34716000 & 15 & 0 & 0 & 0 & 0 \\ -594165 & 0 & 30 & 0 & 0 & 0 \\ -34715963 & -3 & -7 & 1 & 0 & 0 \\ 17655087 & -12 & -18 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{X,3} = \begin{pmatrix} 43543500 & 0 & 0 & 0 & 0 & 0 \\ -37009500 & 825 & 0 & 0 & 0 & 0 \\ -6534165 & -750 & 30 & 0 & 0 & 0 \\ 37 & -18 & -7 & 1 & 0 & 0 \\ 87 & -42 & -18 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

vii. A basis of $\text{Pic}(X_i)$ inside $\text{Cl}(X_i)$ is then obtained by applying part 6 of Theorem 3.2. For $i = 1, 2, 3$, matrices A_i switching $C_{X_i} \cdot Q^T$ in Hermite normal form are respectively

$$A_1 = \begin{pmatrix} -351039 & -449987 & -449987 & 0 & 0 & 0 \\ -502913 & -644670 & -644670 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -93838 & -117699 & 0 & 0 & 0 & 0 \\ -1157199 & -1451450 & 0 & 0 & 0 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -10317 & -12139 & 0 & 0 & 0 & 0 \\ -22429 & -26390 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

giving

$$B_{X_1} = {}^2(A_1 \cdot C_{X_1} \cdot Q^T) = \begin{pmatrix} 825 & 185620050 \\ 0 & 265926375 \end{pmatrix}$$

$$B_{X_2} = {}^2(A_2 \cdot C_{X_2} \cdot Q^T) = \begin{pmatrix} 60 & 1765515 \\ 0 & 21771750 \end{pmatrix}$$

$$B_{X_3} = {}^2(A_3 \cdot C_{X_3} \cdot Q^T) = \begin{pmatrix} 3300 & 10016325 \\ 0 & 21771750 \end{pmatrix}$$

$$\Theta_{X_i} = {}^2(A_i \cdot C_{X_i} \cdot \Gamma^T) = \begin{pmatrix} [0]_3 & [0]_{15} \\ [0]_3 & [0]_{15} \end{pmatrix}, \quad \text{for } i = 1, 2, 3.$$

References

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2. Rossi, M., and Terracini, L. *A \mathbb{Q} -factorial complete toric variety is a quotient of a poly weighted space* *Ann. Mat. Pur. Appl.* **196** (2017), 325–347; DOI:10.1007/s10231-016-0574-7; [arXiv:1502.00879](#).